Algebra II — Lesson 0504
Dr. Paul L. Bailey — Wednesday, May 4, 2022

1. Review

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes. An *event* is a subset of the sample space.

The *cardinality* of a set is the number of things in it. The cardinality of the set A is denoted $|A|$.

The *probability* of event E is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

For a positive integer n, we define n factorial as the product of all positive integers less than or equal to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

This is the number of ways of rearranging n things.

The number of *permutations* (ordered subsets) of n things taken k at a time is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The number of *combinations* (unordered subsets) of n things taken k at a time is

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

The number of combinations is typically referred to as "n choose k", and is also written

$$\binom{n}{k} = C(n,k).$$

2. Motivational Example

Example 1. We roll a die three times. What is the probability that all of the values are less than or equal to three?

Solution. One outcome is a sequence of three digits between one and six. We model this using ordered triples from the set $N_6 = \{1,2,3,4,5,6\}$. Thus, the sample space is

$$S = \{(a,b,c) \mid a,b,c \in N_6\}.$$

Then

$$|S| = 6^3 = 216.$$

The event E is the subset of S consisting of ordered triples from the set $N_3 = \{1,2,3\}$; that is,

$$E = \{(a,b,c) \mid a,b,c \in N_3\}.$$

So

$$|E| = 3^3 = 27.$$

Thus

$$P(E) = \frac{27}{216} = 0.125.$$

□

3. Cartesian Product

An *ordered n-tuple* is a sequence of n elements, typically written surrounded by parentheses. For example, $(2, 7)$ is an ordered pair, $(5, 2, 9)$ is an ordered triple, and so forth.

Let A and B be sets. The *cartesian product* of A and B is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

So, $A \times B$ is the set of ordered pairs whose first entry is from A and whose second entry is from B.

Let A, B, and C be sets. The *cartesian product* of A, B, and C is

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, \text{ and } c \in C\}.$$

So, $A \times B \times C$ is the set of ordered triples whose first entry is from A, whose second entry is from B, and whose third entry is from C.

More generally, if $A_1, A_2, \ldots, A_n$ is a list of n sets, the *cartesian product* of these sets is the set of ordered n-tuples whose i^{th} entry is from the set A_i, for $i = 1, 2, \ldots, n$.

Consider $A \times B$. We have $|A|$ choices for the first entry, and $|B|$ choices for the second entry. Together, we see that

$$|A \times B| = |A| \cdot |B|.$$

More generally, the cardinality of the cartesian product of n sets is the product of the cartesian products of the given sets.

Example 2. If we flip a coin five times, what is the probability the we get exactly four heads?

Solution. Let $T = \{0, 1\}$ denote the set of outcomes of flipping the coin once. We assign 0 to tails and 1 to heads.

The sample space of this problem is the set of ordered quintuples from T; that is,

$$S = T \times T \times T \times T \times T = \{(a, b, c, d, e) \mid a, b, c, d, e \in T\}.$$

Thus

$$|S| = 2^5 = 32.$$

The event E is the set of ordered quintuples with exactly four ones:

$$E = \{(1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1)\}.$$

Thus $|E| = 5$, and

$$P(E) = \frac{5}{32} = 0.15625.$$

□

Example 3. We have 20 balls in a bin. There are 5 red balls, 7 white balls, and 8 blue balls. We select three distinct balls at random.

(a) Find the probability that all the balls chosen were blue.
(b) Find the probability of choosing one ball of each color.

Solution. The sample space S is the set of sets of three balls, and

$$|S| = \binom{20}{3} = 1140.$$

(a) In this part, the event E is the set of sets of three blue balls. The size of E is the number of ways of choosing 3 balls from a set of 8 balls; that is,

$$|E| = \binom{8}{3} = 42.$$

So,

$$P(E) = \frac{42}{1140} = 0.0368.$$

(b) In this part, the event E is the set of sets of three balls, one of each color. This set may be viewed as the cartesian product of the sets of red, white, and blue balls, so the size of E is the product of these sizes:

$$|E| = 5 \times 7 \times 8 = 280.$$

So

$$P(E) = \frac{280}{1140} = 0.2456.$$

□